大阪学院大学経済論集
第30巻第１・２号、2016年６月

Stochastic Properties of Unit Root Tests under a Stationarity Alternative with Multiple Structural Breaks

Takashi Matsuki

ABSTRACT

This study investigates the stochastic properties of the Dickey–Fuller t-test and $T(\hat{\rho}-1)$ test for multiple structural breaks（in level or slope）in the trend function of a stationary time series.　In the presence of H（≥ 2）breaks in the series, the asymptotic analysis and Monte Carlo simulation indicate some common features of the tests that are consistent with previous studies and produce some new results as well.

Keywords：Unit roots; Dickey–Fuller test; Multiple breaks; Stationarity; Fewer rejection; Perron phenomenon.

JEL Classification Numbers：C12; C15; C22.

1. Introduction

Many papers have been published on unit root tests with structural breaks since the studies by Perron（1989）and Rappoport and Reichlin（1989）.[1] Perron（1989）demonstrated that there can be fewer rejections of the unit root null hypothesis in the Dickey－Fuller（DF）test, which was proposed by Dickey and Fuller（1979）, when a series is generated by a stationary process with a break（also known as the "Perron phenomenon"）. With regard to this problem, Montanes and Reyes（1998, 1999）have examined the asymptotic behavior of the DF t-statistic and $T(\hat{\rho}-1)$ statistic under the alternative hypotheses of "changing growth" and "crash"（i.e., the stationarity hypotheses with shifts in slope and level）. Leybourne and Newbold（2000）reported that the "Perron phenomenon" becomes more severe in the DF t-test when there is a single break within a specific range of a sample. Furthermore, Sen（2001）studied how the presence of one break in a stationary process affects Dickey and Fuller's（1981）F-test.[2]

However, all these previous studies examined only the effects of the presence of a single break in a series on the hypothesis test.[3] Therefore, in this study, we

1）For example, see Banerjee, Lumsdaine and Stock（1992）and Zivot and Andrews（1992）for the case of an unknown single break and Lumsdaine and Papell（1997）and Lee and Strazicich（2003）for the case of unknown multiple breaks. On the other hand, Lee（1999）and Becker, Enders and Lee（2006）have developed the stationarity tests with multiple structural breaks.

2）Leybourne, Mills and Newbold（1998）and Lee（2000）have discussed the spurious rejection problem of the DF t-test leading to the possible over-rejection of the unit root null hypothesis when the data generating process is integrated of order one with a break.

3）Some Japanese macroeconomic time series are suspected to have multiple structural breaks; for example, real and nominal GDP, private and household consumption expenditures, and M2 + CD. If these series actually have multiple breaks, they will not be dealt with in the same framework as the earlier studies.

generated a time series, y_t, using the following model to analyze the case with multiple structural breaks.

$$y_t = \alpha + \beta t + d_t + z_t,$$

$$d_t = Scale \sum_{h=1}^{H} k_h Dummy_t^h,$$

$$z_t = \rho z_{t-1} + \varepsilon_t,\ |\rho| < 1,\ t = 0,\ldots,T, \qquad (1)$$

where $Scale$ is the scale factor of the dummy variables, H is the maximum number of breaks, ε_t is $i.i.d.(0,\sigma^2)$, and $T+1$ is the sample size. If h denotes the order of a break ($h = 1,\ldots,H$), then k_h is the size of the h^{th} break and $Dummy_t^h$ is its dummy variable. The structural breaks are considered as shifts in level or shifts in slope. For shifts in level, $Dummy_t^h$ is defined as $Dummy_t^h = DU_t^h$, where $DU_t^h = 1$ for $t > \tau_h T$ and 0 otherwise. τ_h is the h^{th} break fraction, which is defined as TB_h / T for all T (TB_h is the h^{th} break point), and $0 < \tau_1 < \tau_2 < \cdots < \tau_{H-1} < \tau_H < 1$. For shifts in slope, $Dummy_t^h$ is defined as $Dummy_t^h = DT_t^h$, where $DT_t^h = t - \tau_h T$ for $t > \tau_h T$ and 0 otherwise.

By changing the value of the scale factor of the dummy variable $Scale$ in the above model, the various models used in previous studies can be expressed. For example, when $Scale$ takes a value of one, the model becomes the "crash" alternative model for $H = 1$ (a single break) and $Dummy_t = DU_t$ (a shift in level), and the "changing growth" alternative model for $H = 1$ and $Dummy_t = DT_t$ (a shift in slope), both of which have been used in Montanes and Reyes (1998, 1999). When $Scale$ takes the values $T^{1/2}$, with $H = 1$ and $Dummy_t = DU_t$ (a shift in level), and $T^{-1/2}$, with $H = 1$ and $Dummy_t = DT_t$ (a shift in slope) in model (1), the consequent models are consistent with the ones that have been assumed by Leybourne and Newbold (2000).

The objective of this study is to investigate the effects of the presence of multiple structural breaks in a stationary process on the test of the unit root hypotheses of the DF t-test and $T(\hat{\rho}-1)$ test.　In the next section, the limiting distributions of the statistics are derived under the model with breaks and the asymptotic behavior of the statistics is analyzed for various locations and sizes of breaks.　In Section 3, a Monte Carlo simulation is conducted to examine the empirical powers of the tests in small samples.　The conclusions are provided in Section 4.

2. Limiting Behavior of the Dickey–Fuller Tests

2.1. Limiting distributions of the Dickey–Fuller statistics under the model with multiple structural breaks

The limiting distributions of the DF t-statistic and $T(\hat{\rho}-1)$ statistic are derived under model (1), which has multiple (H) structural breaks.　In this derivation, *Scale* takes values of either one or $T^{1/2}$ for shifts in level and one or $T^{-1/2}$ for shifts in slope.

The test statistics are obtained in the following way.　Let e_{t-1} denote the residual of the regression of y_{t-1} on an intercept and time trend, for $t = 1,\dots,T$.　Then, the first difference of e_t is regressed as follows:

$$\Delta e_t = \phi e_{t-1} + error,\ t = 1,\dots,T. \tag{2}$$

The DF t-statistic is obtained as a usual t-statistic test of the null hypothesis $\phi = 0$ and the DF $T(\hat{\rho}-1)$ statistic is given by $T\hat{\phi}$, where $\hat{\phi}$ ($=\hat{\rho}-1$) is the estimated coefficient of e_{t-1} in (2).　The limiting distributions of the statistics are indicated by the following two theorems.

Theorem 1. *Under model (1) with* $Scale = 1$,

(a) For shifts in level (the "crash" alternative model in the case of multiple breaks),

$$T^{-1/2} t \xrightarrow{p} -\sigma \sqrt{\frac{1-\rho}{(1+\rho)\{\sigma^2 + 2(1-\rho)b_{11}\}}}, \tag{3}$$

$$T^{-1}\{T(\hat{\rho}-1)\} \xrightarrow{p} -\frac{\sigma^2(1-\rho)}{\sigma^2 + (1-\rho^2)b_{11}}, \tag{4}$$

(b) For shifts in slope (the "changing growth" alternative model in the case of multiple breaks), such that at least one $k_h \neq 0$ $(h = 1, \ldots, H)$,

$$T^{-1/2} t \xrightarrow{p} \frac{b_{21}}{\sqrt{\left(\frac{2\sigma^2}{1+\rho} + b_{22}\right) b_{23} - b_{21}^2}}, \tag{5}$$

$$T(\hat{\rho}-1) \xrightarrow{p} \frac{b_{21}}{b_{23}}, \tag{6}$$

where $\xrightarrow{p}$ *represents convergence in probability.* b_{11}, b_{21}, b_{22}, *and* b_{23} *are given by*

$$b_{11} = \sum_{h=1}^{H} k_h^2 \tau_h (1-\tau_h)(3\tau_h^2 - 3\tau_h + 1) + 2\sum_{g=1}^{H-1}\sum_{h=2}^{H} k_g k_h \tau_g (1-\tau_h)(3\tau_g \tau_h - 3\tau_h + 1), \quad \text{where } g < h,$$

$$b_{21} = \frac{1}{2}\sum_{g=1}^{H}\sum_{h=1}^{H} k_g k_h \tau_g \tau_h (1-\tau_g)(1-\tau_h)(\tau_g + \tau_h - 1),$$

$$b_{22} = \sum_{h=1}^{H} k_h^2 \tau_h (1-\tau_h)\left\{\tau_h^2 (2\tau_h - 3)(1-2\tau_h) + (1-\tau_h)(2\tau_h + 1)\right\},$$

$$+ 2\sum_{g=1}^{H-1}\sum_{h=2}^{H} k_g k_h \tau_g (1-\tau_h)\left\{\tau_g \tau_h (2\tau_g - 3)(1-2\tau_h) + (1-\tau_h)(2\tau_h + 1)\right\}, \quad \text{where } g < h,$$

$$b_{23}=\frac{1}{3}\left\{\sum_{h=1}^{H}k_h^2\tau_h^3(1-\tau_h)^3-\sum_{g=1}^{H-1}\sum_{h=2}^{H}k_gk_h\tau_g^2(1-\tau_h)^2(2\tau_g\tau_h+\tau_g-3\tau_h)\right\},\ \textit{where}\ g<h.$$

The proof is demonstrated in the Appendix.

Theorem 2. *Under model (1) with* $Scale=T^{1/2}$ *for shifts in level and* $Scale=T^{-1/2}$ *for shifts in slope, such that at least one* $k_h\neq 0$ $(h=1,\ldots,H)$,

$$t\xrightarrow{p}\frac{-\dfrac{\sigma^2}{1+\rho}+c_{i1}}{\sqrt{\left(\dfrac{2\sigma^2}{1+\rho}+c_{i2}\right)c_{i3}}},\ i=1,2, \tag{7}$$

$$T(\hat{\rho}-1)\xrightarrow{p}\frac{-\dfrac{\sigma^2}{1+\rho}+c_{i1}}{c_{i3}},\ i=1,2, \tag{8}$$

where the subscripts $i=1$ *and* 2 *denote* H *times shifts in level and slope, respectively.* c_{i1}, c_{i2}, *and* c_{i3} *are given by*

$$c_{11}=-\sum_{h=1}^{H}k_h^2(1-\tau_h)(6\tau_h^2-3\tau_h+1)$$
$$-\sum_{g=1}^{H-1}\sum_{h=2}^{H}k_gk_h\{(1-\tau_g)(6\tau_g\tau_h-3\tau_g+1)+(1-\tau_h)(6\tau_g\tau_h-3\tau_h+1)-1\},\ \textit{where}\ g<h,$$

$$c_{12}=\sum_{h=1}^{H}k_h^2,\ c_{13}=b_{11},\ c_{21}=b_{21},\ c_{22}=0,\ \textit{and}\ c_{23}=b_{23}.$$

The proof is demonstrated in the Appendix.

These theorems have several implications. For $Scale = 1$, Theorem 1(a) suggests that for shifts in level, as $T \to \infty$, the t-statistic and the $T(\hat{\rho}-1)$ statistic diverge to $-\infty$ at rates of $T^{1/2}$ and T, respectively. Thus, both the DF tests are consistent despite the presence of multiple breaks in the series. This fact corresponds to Proposition 1 in Montanes and Reyes (1999). In Theorem 1(b), for shifts in slope, the t-statistic diverges at a rate of $T^{1/2}$, whereas the $T(\hat{\rho}-1)$ statistic converges in probability to a nonrandom limiting function of locations (τ_h) and sizes (k_h) of breaks. For $Scale = T^{1/2}$ for shifts in level and $Scale = T^{-1/2}$ for shifts in slope, Theorem 2 indicates that both the statistics converge to nonrandom limiting functions of 2(H+1) parameters: ρ, σ, τ_h, and k_h.

2.2. Effects of two breaks on the Dickey–Fuller tests

In this subsection, the effects of the presence of two breaks on the DF t-test and $T(\hat{\rho}-1)$ test for large samples are considered. Tables 1-3 report the limiting distributions of the tests for two shifts in level and slope each ($H = 2$). The values in the table are computed in the region $0 < \tau_1 < \tau_2 < 1$ at 0.01 intervals with $\rho = 0.9$ and $\sigma = 1$. Then, the sizes of two breaks, k_1, k_2, take the following values: (0.25, 0.25), (1.0, 0.25), (0.25, 1.0), and (1.0, 1.0) for shifts in level and (5, 5), (20, 5), (5, 20), and (20, 20) for shifts in slope.[4)]

For breaks in slope with $Scale = 1$, Table 1 presents the limiting distributions of the t-statistic multiplied by the $T^{-1/2}$ and the $T(\hat{\rho}-1)$ statistic. For $T^{-1/2}t$, approximately for $\tau_1 \geq 0.5$ or $\tau_2 = 0.71$, its limiting distribution takes positive

4) We have also analyzed some cases where one of the two break sizes takes negative values: (−0.25, 0.25), (−1.0, 0.25), and (0.25, −1.0) for shifts in level and (−5, 5), (−20, 5), and (5, −20) for shifts in slope. Consequently, for any combination of break sizes, both the DF statistics indicate behaviors similar to those observed in Table 1(a), 1(b), and 1(c) when the absolute value of the break size, $|k_h|$, increases. Therefore, the results are omitted here.

values, which implies that there would be fewer rejections of the unit root null hypothesis in these regions in the usual DF t-test with breaks as the t-statistic（t）tends to $+\infty$ as $T \to \infty$.　For the $T(\hat{\rho}-1)$ test, the statistic takes values higher than the corresponding critical values in all the cases except when two break fractions, (τ_1, τ_2), are close to (0.01,0.02) for each pair of (k_1, k_2) and (0.01,0.99) for the small size of the second break（$k_2 = 5$）.[5)]　This result implies that the $T(\hat{\rho}-1)$ test may fail to reject the unit root null hypothesis in many cases, excluding the specific cases described above.

5）The critical values are -29.4, -21.7, and -18.3 at 1%, 5%, and 10% significance levels, respectively, in Fuller（1996）.

Table 1　The limiting distributions of the DF tests with $Scale = 1$（two shifts in slope）

$T^{-1/2}t$

(k_1, k_2)	τ_1	τ_2 = 0.02	0.11	0.31	0.51	0.71	0.91	0.99
(5, 5)	0.01	-0.88	-0.79	-0.35	-0.02	0.31	0.61	0.00
	0.1		-0.83	-0.54	-0.21	0.06	-0.04	-0.61
	0.3			-0.35	-0.17	0.01	-0.06	-0.29
	0.5				0.01	0.19	0.21	0.03
	0.7					0.36	0.56	0.37
	0.9						0.86	0.81
(20, 5)	0.01	-1.48	-0.93	-0.41	-0.10	0.17	0.24	-0.89
	0.1		-0.90	-0.80	-0.58	-0.48	-0.70	-0.87
	0.3			-0.36	-0.30	-0.25	-0.29	-0.35
	0.5				0.00	0.07	0.06	0.01
	0.7					0.37	0.43	0.37
	0.9						0.92	0.94
(5, 20)	0.01	-1.41	-0.89	-0.35	0.01	0.37	0.91	0.89
	0.1		-0.88	-0.41	-0.05	0.31	0.68	-0.26
	0.3			-0.35	-0.05	0.27	0.46	-0.15
	0.5				0.01	0.32	0.58	0.11
	0.7					0.38	0.83	0.43
	0.9						0.95	0.99
(20, 20)	0.01	-1.79	-0.99	-0.38	-0.02	0.34	0.78	0.00
	0.1		-0.90	-0.57	-0.22	0.06	-0.04	-0.76
	0.3			-0.35	-0.17	0.01	-0.07	-0.32
	0.5				0.01	0.19	0.22	0.03
	0.7					0.37	0.59	0.40
	0.9						0.95	1.04

$T(\hat{\rho}-1)$ test[1)]

(k_1, k_2)	τ_1	τ_2 = 0.02	0.06	0.11	0.21	0.51	0.91	0.96	0.99
(5, 5)	0.01	-95.14	-30.22	-14.43	-6.07	-0.12	14.32	31.98	0.00
	0.05		-25.69	-16.46	-8.02	-1.02	6.68	-5.47	-26.14
	0.1			-12.61	-7.74	-1.75	-0.85	-8.54	-12.74
	0.2				-5.43	-1.83	-1.96	-4.20	-5.36
	0.5					0.06	1.77	0.94	0.24
	0.9						14.14	18.59	16.25
	0.98								95.14
(20, 5)	0.01	-121.5	-50.02	-22.53	-8.79	-0.93	9.01	-5.50	-130.1
	0.05		-27.27	-22.33	-14.60	-5.28	-17.12	-25.61	-28.14
	0.1			-13.04	-10.67	-6.47	-11.25	-12.70	-13.23
	0.2				-5.55	-4.07	-4.98	-5.36	-5.56
	0.5					0.02	0.42	0.22	0.06
	0.9						13.65	14.83	14.05
	0.98								80.10
(5, 20)	0.01	-80.10	-25.10	-12.56	-5.44	0.06	14.90	35.52	130.1
	0.05		-24.27	-13.29	-5.95	-0.14	13.96	28.13	-4.10
	0.1			-12.21	-6.01	-0.32	11.94	14.07	-8.54
	0.2				-5.32	-0.45	8.05	3.81	-4.13
	0.5					0.10	6.63	4.55	1.07
	0.9						14.66	26.50	25.71
	0.98								121.5

1) The result for $(k_1, k_2) = (20, 20)$ is omitted in this table because it is the same as that for $(k_1, k_2) = (5, 5)$.

For $Scale = T^{1/2}$ for shifts in level and $Scale = T^{-1/2}$ for shifts in slope, the results of the t-test are presented in Tables 2 and 3, respectively.　For shifts in level, Table 2 indicates that as k_1 or k_2 increase(s), almost all the probability limits tend to rise.[6)]　Consequently, this climb in the values of the limiting distribution can cause the "Perron phenomenon." For shifts in slope, Table 3 indicates that the limiting value of the statistic becomes as low as, $(\tau_1, \tau_2) \to (0,0)$, $(1,1)$, or $(0,1)$.　However, besides these three regions, we expect fewer rejections of the unit root null hypothesis, particularly in the regions $\tau_1 \leq 0.9$ and $0.51 \leq \tau_2 \leq 0.91$.　The results of the $T(\hat{\rho}-1)$ test for shifts in level and slope are omitted in this paper as the characteristic of the limiting distribution of the test is analogous to that of the t-test for both.

6) For $\tau_1 = 0.01$ and $0.11 \leq \tau_2 \leq 0.91$, the limits tend to decline as k_1 increases.

Table 2　The limiting distributions of the DF tests with $Scale = T^{1/2}$（two shifts in level）

(k_1, k_2)	τ_1	t-test τ_2 0.02	0.11	0.31	0.51	0.71	0.91	0.99
(0.25, 0.25)	0.01	-11.96	-8.12	-7.69	-8.57	-7.74	-8.41	-15.82
	0.1		-4.57	-5.40	-7.17	-7.22	-6.81	-7.98
	0.3			-3.88	-6.19	-8.15	-7.18	-7.47
	0.5				-4.43	-6.31	-7.18	-8.18
	0.7					-4.03	-5.30	-6.92
	0.9						-4.02	-6.35
	0.98							-8.18
(1.0, 0.25)	0.01	-9.79	-8.83	-8.66	-9.04	-8.73	-8.98	-10.19
	0.1		-2.89	-3.07	-3.33	-3.44	-3.38	-3.36
	0.3			-1.95	-2.44	-2.81	-2.73	-2.62
	0.5				-2.36	-2.98	-3.12	-3.01
	0.7					-2.27	-2.56	-2.61
	0.9						-1.67	-1.81
	0.98							-1.72
(0.25, 1.0)	0.01	-7.74	-3.54	-2.57	-2.84	-2.61	-2.34	-4.28
	0.1		-2.80	-2.36	-2.94	-2.79	-2.44	-3.91
	0.3			-1.95	-2.85	-3.05	-2.65	-4.08
	0.5				-2.37	-2.79	-2.58	-4.08
	0.7					-2.25	-2.18	-3.30
	0.9						-1.65	-2.30
	0.98							-1.94
(1.0, 1.0)	0.01	-8.75	-5.16	-3.82	-3.84	-3.45	-3.60	-6.37
	0.1		-2.49	-2.30	-2.86	-2.97	-2.73	-2.99
	0.3			-1.38	-2.29	-3.29	-2.81	-2.68
	0.5				-1.79	-2.75	-2.88	-2.89
	0.7					-1.80	-1.95	-2.03
	0.9						-0.86	-0.54
	0.98							0.64

Table 3　The limiting distributions of the DF tests with $Scale = T^{-1/2}$（two shifts in slope）

(k_1, k_2)	τ_1	t-test τ_2 0.02	0.11	0.31	0.51	0.71	0.91	0.99
(5, 5)	0.01	-49.77	-6.98	-2.66	-1.46	-1.15	-6.61	-126.6
	0.1		-5.13	-3.14	-1.96	-1.52	-4.72	-7.54
	0.3			-2.43	-1.56	-1.02	-1.88	-2.56
	0.5				-0.67	-0.05	-0.61	-1.33
	0.7					0.64	0.08	-0.94
	0.9						-1.52	-5.22
	0.98							-47.82
(20, 5)	0.01	-29.60	-7.53	-3.07	-1.74	-1.37	-6.81	-45.09
	0.1		-6.37	-5.64	-4.53	-4.03	-5.17	-5.67
	0.3			-4.20	-3.44	-2.86	-3.12	-3.50
	0.5				-0.24	0.45	0.25	-0.27
	0.7					3.53	3.47	2.76
	0.9						3.69	2.65
	0.98							-12.32
(5, 20)	0.01	-17.66	-5.79	-3.54	-0.27	2.69	2.04	-42.11
	0.1		-6.32	-4.25	-0.81	2.29	1.66	-7.13
	0.3			-4.10	-0.87	2.21	1.57	-2.33
	0.5				-0.12	2.88	2.55	-1.04
	0.7					3.61	3.59	-0.50
	0.9						3.50	-4.05
	0.98							-25.17
(20, 20)	0.01	-16.10	-6.47	-3.92	-0.52	2.52	1.90	-31.65
	0.1		-8.95	-6.98	-3.07	0.27	-1.52	-5.52
	0.3			-6.29	-3.11	-0.16	-1.20	-3.32
	0.5				-0.01	2.97	2.42	-0.01
	0.7					6.08	6.19	3.15
	0.9						7.13	3.40
	0.98							-8.30

3. Monte Carlo Analysis

To investigate the behavior of the DF t-statistic and $T(\hat{\rho}-1)$ statistic in finite samples, a Monte Carlo simulation is implemented in this section. The series, y_t, is generated by model (1) with $H=2$, $\alpha=\beta=1$, and $\rho=0.9$, where *Scale* takes values of either one or $T^{1/2}$ for shifts in level and one or $T^{-1/2}$ for shifts in slope. Further, ε_t is $i.i.d.N(0,1)$. The sample size is 200 and the number of replications is 5000. The empirical powers of both the DF tests at the nominal level (5%) are computed over the same combinations of (τ_1,τ_2) and (k_1,k_2) as in Tables 1-3.[7)] In this simulation, when there is no break in the series, the arithmetic means of the empirical powers of the t-test and the $T(\hat{\rho}-1)$ test at 5% significance level are about 66% and 70%, respectively. To evaluate the empirical powers in the presence of two breaks, we treat these means as baseline values following Leybourne and Newbold (2000).

The results of the shifts in level model with $Scale=1$, which are not reported here, indicate that all the empirical powers in both the DF tests are very close to their baseline values. Thus, the tests are not affected by the presence of breaks even in small samples. As regards the case of shifts in slope with $Scale=1$, Table 4 presents the experimental results of the tests. There are fewer rejections of the unit root null hypothesis for $\tau_1 \geq 0.5$ or $0.51 \leq \tau_2 \leq 0.91$, excluding the case of $(k_1,k_2)=(20{,}5)$ in the t-test, and for all the combinations of (τ_1,τ_2), excluding $(\tau_1,\tau_2)=(0.01{,}0.02)$ and $(0.01{,}0.99)$ in the $T(\hat{\rho}-1)$ test. The results of Table 1 in

7) In addition, we have conducted the simulation for the same cases of negative break sizes as those described in footnote 4. For all the combinations of these break sizes, the obtained results are similar to those in Tables (4), (5), and (6), except some cases in the shifts in slope model. As $|k_2|$ becomes large, the empirical power increases around $(\tau_1,\tau_2)=(0.98{,}0.99)$ for the t-test with $Scale=T^{-1/2}$ and for the $T(\hat{\rho}-1)$ test with $Scale=1$ and $T^{-1/2}$.

the previous section predict this failure to reject the null hypothesis in the tests. Therefore, the results obtained in small samples precisely correspond to those in large samples, as indicated in Table 1.

Table 4　The empirical powers of the DF tests with $Scale = 1$（two shifts in slope）

t-test

(k_1, k_2)	τ_1	τ_2 = 0.02	0.11	0.31	0.51	0.71	0.91	0.99
(5, 5)	0.01	100.0	100.0	100.0	0.00	0.00	0.00	37.64
	0.1		100.0	100.0	2.22	0.00	0.00	100.0
	0.3			100.0	0.00	0.00	0.00	100.0
	0.5				0.00	0.00	0.00	0.00
	0.7					0.00	0.00	0.00
	0.9						0.00	0.00
(20, 5)	0.01	100.0	100.0	100.0	0.00	0.00	0.00	100.0
	0.1		100.0	100.0	100.0	100.0	100.0	100.0
	0.3			100.0	100.0	100.0	100.0	100.0
	0.5				0.00	0.00	0.00	0.00
	0.7					0.00	0.00	0.00
	0.9						0.00	0.00
(5, 20)	0.01	100.0	100.0	100.0	0.00	0.00	0.00	0.00
	0.1		100.0	100.0	0.00	0.00	0.00	100.0
	0.3			100.0	0.00	0.00	0.00	0.00
	0.5				0.00	0.00	0.00	0.00
	0.7					0.00	0.00	0.00
	0.9						0.00	0.00
(20, 20)	0.01	100.0	100.0	100.0	0.00	0.00	0.00	1.20
	0.1		100.0	100.0	0.00	0.00	0.00	100.0
	0.3			100.0	0.00	0.00	0.00	100.0
	0.5				0.00	0.00	0.00	0.00
	0.7					0.00	0.00	0.00
	0.9						0.00	0.00

$T(\hat{\rho}-1)$ test

(k_1, k_2)	τ_1	τ_2 = 0.02	0.06	0.11	0.21	0.51	0.91	0.96	0.99
(5, 5)	0.01	100.0	100.0	0.00	0.00	0.00	0.00	0.00	63.00
	0.05		100.0	0.00	0.00	0.00	0.00	0.00	99.86
	0.1			0.00	0.00	0.00	0.00	0.00	0.00
	0.2				0.00	0.00	0.00	0.00	0.00
	0.5					0.00	0.00	0.00	0.00
	0.9						0.00	0.00	0.00
	0.98								0.00
(20, 5)	0.01	100.0	100.0	0.02	0.00	0.00	0.00	0.00	100.0
	0.05		100.0	100.0	0.00	0.00	0.00	100.0	100.0
	0.1			0.00	0.00	0.00	0.00	0.00	0.00
	0.2				0.00	0.00	0.00	0.00	0.00
	0.5					0.00	0.00	0.00	0.00
	0.9						0.00	0.00	0.00
	0.98								0.00
(5, 20)	0.01	100.0	100.0	0.00	0.00	0.00	0.00	0.00	0.00
	0.05		100.0	0.00	0.00	0.00	0.00	0.00	4.82
	0.1			0.00	0.00	0.00	0.00	0.00	0.00
	0.2				0.00	0.00	0.00	0.00	0.00
	0.5					0.00	0.00	0.00	0.00
	0.9						0.00	0.00	0.00
	0.98								0.00
(20, 20)	0.01	100.0	100.0	0.00	0.00	0.00	0.00	0.00	36.94
	0.05		100.0	0.00	0.00	0.00	0.00	0.00	100.0
	0.1			0.00	0.00	0.00	0.00	0.00	0.00
	0.2				0.00	0.00	0.00	0.00	0.00
	0.5					0.00	0.00	0.00	0.00
	0.9						0.00	0.00	0.00
	0.98								0.00

The results for shifts in level for $Scale = T^{1/2}$ are presented in Table 5. In both the DF tests, when two breaks are in close proximity, except for two early breaks, the tests are seriously biased in favor of fewer rejections of the unit root null hypothesis. As k_1 or k_2 increase(s), the biases of the tests also become large, except in some extreme cases of (τ_1, τ_2).[8)] Table 6 indicates the empirical powers in the shifts in slope model with $Scale = T^{-1/2}$. When we focus on the regions $0.3 \le \tau_1 \le 0.9$ or $0.51 \le \tau_2 \le 0.91$ for the t-test and $0.1 \le \tau_1 \le 0.9$ or $0.11 \le \tau_2 \le 0.91$ for the $T(\hat{\rho} - 1)$ test, the rejection frequencies within these specific regions of both the tests exhibit extremely few rejections of the null hypothesis.

8) The empirical powers increase as $\tau_1 \to 0$ for the case of $(k_1, k_2) = (1.0, 0.25)$ (the case with large k_1), $(\tau_1, \tau_2) \to (0,0)$ for the case of $(0.25, 1.0)$ (the case with large k_2), and $(\tau_1, \tau_2) \to (0,1)$ for the case of $(1.0, 1.0)$ (the case with large break sizes in both).

Table 5　The empirical powers of the DF tests with $Scale = T^{1/2}$ (two shifts in level)

t-test

(k_1, k_2)	τ_1	τ_2 0.02	0.06	0.11	0.31	0.51	0.91	0.99
(0.25, 0.25)	0.01	93.72	87.96	80.10	63.70	63.68	62.12	73.56
	0.05		62.84	54.04	49.62	56.88	55.22	63.68
	0.1			26.28	36.76	52.28	48.54	54.34
	0.3				12.20	41.30	50.62	49.72
	0.5					20.86	50.86	53.28
	0.9						12.80	30.24
	0.98							29.10
(1.0, 0.25)	0.01	100.0	100.0	100.0	100.0	100.0	99.94	100.0
	0.05		90.32	84.72	82.46	88.04	90.62	93.50
	0.1			0.66	3.90	10.32	10.60	11.44
	0.3				0.00	0.26	1.12	0.56
	0.5					0.02	5.02	3.26
	0.9						0.00	0.06
	0.98							0.22
(0.25, 1.0)	0.01	100.0	98.16	36.38	0.66	1.78	0.46	19.06
	0.05		74.04	9.20	0.26	1.96	0.22	15.72
	0.1			0.78	0.08	2.88	0.54	14.80
	0.3				0.00	2.42	1.44	17.46
	0.5					0.02	1.00	16.60
	0.9						0.00	2.76
	0.98							0.76
(1.0, 1.0)	0.01	100.0	100.0	100.0	63.32	54.44	40.02	96.90
	0.05		94.30	42.50	2.72	12.70	6.42	46.48
	0.1			0.00	0.00	1.76	0.56	4.20
	0.3				0.00	0.04	1.30	0.72
	0.5					0.00	1.60	2.78
	0.9						0.00	0.00
	0.98							0.00

$T(\hat{\rho}-1)$ test

(k_1, k_2)	τ_1	τ_2 0.02	0.06	0.11	0.31	0.51	0.91	0.99
(0.25, 0.25)	0.01	87.68	79.16	70.32	61.42	64.44	63.66	77.28
	0.05		48.40	42.56	48.16	58.56	56.44	67.44
	0.1			20.08	37.46	55.14	51.06	58.56
	0.3				13.86	45.16	54.58	55.02
	0.5					21.94	54.10	59.78
	0.9						15.74	42.18
	0.98							47.76
(1.0, 0.25)	0.01	99.98	100.0	99.98	99.72	99.74	99.78	100.0
	0.05		14.24	18.30	34.08	53.78	50.78	55.82
	0.1			0.02	0.96	3.64	3.76	3.20
	0.3				0.00	0.38	1.30	0.64
	0.5					0.00	4.02	2.96
	0.9						0.00	0.18
	0.98							4.54
(0.25, 1.0)	0.01	100.0	49.92	4.80	0.56	2.20	0.88	59.16
	0.05		5.80	0.62	0.26	2.58	0.66	54.34
	0.1			0.00	0.10	3.58	1.24	50.54
	0.3				0.00	2.58	2.54	55.12
	0.5					0.04	1.64	54.06
	0.9						0.00	19.70
	0.98							13.28
(1.0, 1.0)	0.01	100.0	99.86	79.24	24.34	35.94	29.40	99.44
	0.05		0.02	0.00	0.20	7.38	3.12	56.76
	0.1			0.00	0.00	1.68	0.44	6.48
	0.3				0.00	0.06	1.62	2.46
	0.5					0.00	1.40	8.18
	0.9						0.00	0.06
	0.98							0.02

Table 6　The empirical powers of the DF tests with $Scale = T^{-1/2}$（two shifts in slope）

t-test

(k_1, k_2)	τ_1	τ_2 = 0.02	0.11	0.21	0.31	0.51	0.71	0.91	0.99
(5, 5)	0.01	68.66	73.14	24.52	1.18	0.00	0.00	24.86	66.16
	0.1		88.30	58.18	13.56	0.06	0.00	28.00	69.76
	0.2			26.54	4.42	0.00	0.00	4.70	22.88
	0.3				0.20	0.00	0.00	0.10	0.84
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.32	14.94
	0.98								62.52
(20, 5)	0.01	73.68	83.80	39.54	3.36	0.00	0.00	28.26	68.56
	0.1		100.0	100.0	100.0	94.20	67.46	98.74	99.98
	0.2			99.98	100.0	95.22	67.22	92.14	99.40
	0.3				64.24	6.44	0.30	3.04	18.40
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.00	0.00
	0.98								29.58
(5, 20)	0.01	83.18	100.0	99.46	15.56	0.00	0.00	0.00	60.14
	0.1		100.0	100.0	81.58	0.00	0.00	0.00	61.44
	0.2			99.98	83.88	0.00	0.00	0.00	15.10
	0.3				53.82	0.00	0.00	0.00	0.54
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.00	6.72
	0.98								50.78
(20, 20)	0.01	88.88	100.0	99.92	29.98	0.00	0.00	0.00	62.44
	0.1		100.0	100.0	100.0	0.06	0.00	0.08	99.98
	0.2			100.0	100.0	2.16	0.00	0.08	97.70
	0.3				99.98	0.00	0.00	0.00	8.44
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.00	0.00
	0.98								15.84

$T(\hat{\rho}-1)$ test

(k_1, k_2)	τ_1	τ_2 = 0.02	0.06	0.11	0.21	0.51	0.71	0.91	0.99
(5, 5)	0.01	71.54	71.22	54.06	3.00	0.00	0.00	33.86	69.60
	0.05		74.00	53.62	3.24	0.00	0.00	33.68	71.52
	0.1			24.26	0.32	0.00	0.00	23.04	57.10
	0.2				0.00	0.00	0.00	0.64	4.64
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.78	24.20
	0.98								67.62
(20, 5)	0.01	74.00	74.50	62.48	4.62	0.00	0.00	36.42	70.96
	0.05		88.16	71.18	9.16	0.00	0.00	53.22	84.44
	0.1			0.10	0.00	0.00	0.00	0.28	1.42
	0.2				0.00	0.00	0.00	0.00	0.00
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.00	0.00
	0.98								46.92
(5, 20)	0.01	79.54	79.28	0.16	0.00	0.00	0.00	0.00	67.74
	0.05		81.76	0.42	0.00	0.00	0.00	0.00	67.08
	0.1			0.04	0.00	0.00	0.00	0.00	52.72
	0.2				0.00	0.00	0.00	0.00	2.90
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.00	14.94
	0.98								61.92
(20, 20)	0.01	84.08	88.16	0.94	0.00	0.00	0.00	0.00	68.96
	0.05		95.30	2.36	0.00	0.00	0.00	0.00	81.66
	0.1			0.00	0.00	0.00	0.00	0.00	1.40
	0.2				0.00	0.00	0.00	0.00	0.00
	0.5					0.00	0.00	0.00	0.00
	0.7						0.00	0.00	0.00
	0.9							0.00	0.00
	0.98								35.16

4. Conclusions

This study investigated how the presence of multiple structural breaks in the stationarity alternative hypothesis affects the DF t-test and $T(\hat{\rho}-1)$ test. In addition, it derived the limiting distributions of the statistics in a model with multiple breaks.

The behavior of the DF statistics was analyzed for two structural breaks in the level or slope of a series over various locations and sizes of the breaks in small as well as large samples. Consequently, some interesting results have been obtained from the asymptotic analysis and the Monte Carlo simulation. The common features of the results are described in the following. For two shifts in level with $Scale=1$ (the "crash" alternative hypothesis in the case of two breaks), both the DF tests are free from the presence of breaks, which Montanes and Reyes (1999) have reported in the single break model. For two shifts in slope with $Scale=1$ (the "changing growth" alternative in the case of two breaks), extremely few rejections of the unit root null can be observed for the first break occurring in the second half of the series or the second break occurring around the 70% point of the series, except for large k_1 and small k_2 in the t-test, and for any location of the two breaks, except for both the breaks occurring at the beginning of the series or the first and second breaks occurring at the beginning and end of the series, respectively, in the $T(\hat{\rho}-1)$ test.

For two shifts in level with $Scale=T^{1/2}$, for large k_1 or k_2, the unit root hypotheses of both the tests would face fewer rejections at all the locations of the two breaks, except for two early breaks and an early (first) break for large k_1 (and any k_2). For two shifts in slope with $Scale=T^{1/2}$, the DF tests might have few powers against the stationarity alternative with breaks, except for cases with two early breaks, early and late breaks, and two late breaks in the series.

Finally, it should be noted that this study demonstrated that extremely few rejections of the unit root null hypothesis can also happen in the DF tests when 2H parameters (locations and magnitudes) of multiple structural breaks take various possible values.

References

Banerjee, A., R.L. Lumsdaine, & J.H. Stock (1992) Recursive and sequential tests of the unit-root and trend-break hypotheses: theory and international evidence. *Journal of Business & Economic Statistics* 10, 271-287.

Becker, R., W. Enders & J. Lee (2006) A stationarity test in the presence of an unknown number of smooth breaks. *Journal of Time Series Analysis* 27, 381-409.

Dickey, D.A. & W.A. Fuller (1979) Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association* 74, 427-431.

Dickey, D.A. & W.A. Fuller (1981) Likelihood ratio statistics for autoregressive time series with a unit root. *Econometrica* 49, 1057-1072.

Fuller, W.A. (1996) *Introduction to Statistical Time Series*, New York: John Wiley & Sons. Lee, J. (1999) Stationarity tests with multiple endogenized breaks. *Non-linear Time Series Analysis of Economic and Financial Data* (In Rothman, P. Ed.), Kluwer Academic Press, 143-163.

Lee, J. (2000) On the end-point issue in unit root tests in the presence of a structural break. *Economics Letters* 68, 7-11.

Leybourne, S.J., T.C. Mills, & P. Newbold (1998) Spurious rejection by Dickey-Fuller tests in the presence of a break under the null. *Journal of Econometrics* 87, 191-203.

Leybourne, S.J. & P. Newbold (2000) Behavior of Dickey-Fuller t-tests when there is a break under the alternative hypothesis. *Econometric Theory* 16, 779-789.

Lumsdaine, R.L & D.H. Papell (1997) Multiple trend breaks and the unit-root hypothesis. *The Review of Economics and Statistics* 79, 212-218.

Montanes, A. & M. Reyes (1998) Effect of a shift in the trend function on Dickey-Fuller unit root tests. *Econometric Theory* 14, 355-363.

Montanes, A. & M. Reyes (1999) The asymptotic behaviour of the Dickey-Fuller tests under the crash hypothesis. *Statistics and Probability Letters* 42, 81-89.

Perron, P. (1989) The great crash, the oil price shock, and the unit root hypothesis.

Econometrica 57, 1361-1401.
Rappoport, P. & L. Reichlin（1989）Segmented trend and non-stationary time series. *The economic Journal* 99, 168-177.
Sen, A.（2001）Behaviour of Dickey-Fuller F-tests under the trend-break stationary alternative. *Statistics and Probability Letters* 55, 257-268.
Zivot, E. & D.W.K. Andrews（1992）Further evidence on the great crash, the oil-price shock, and the unit-root hypothesis. *Journal of Business & Economic Statistics* 10, 251-270.

APPENDIX

The proof of the two theorems is shown for the double breaks case（$H = 2$）because that for the multiple breaks case can be obtained along the same lines but is tedious algebra.

To prove the theorems, some variables are defined in advance.　Let e_{t-1} denote the OLS residuals in the following regression equation.

$$y_{t-1} = \mu + \gamma t + u_t,\ \ t = 1,\ldots,T.$$

e_{t-1} consists of three parts as follows:

$$e_{t-1} = S_{t-1}^{\tau} + g_{t-1} - h_{t-1},$$

where $g_{t-1} = d_{t-1} - T^{-1}\sum_{t=1}^{T} d_{t-1} = d_{t-1} - \bar{d}$ and $h_{t-1} = (t-\bar{t})\sum_{t=1}^{T}(t-\bar{t})d_{t-1}\left\{\sum_{t=1}^{T}(t-\bar{t})^2\right\}^{-1}$.

Also, S_{t-1}^{τ} is

$$S_{t-1}^{\tau} = S_{t-1} - T^{-1}\sum_{t=1}^{T} S_{t-1} - \hat{\delta}(t-\bar{t}),$$

where $S_t = \sum_{s=0}^{\infty}\rho^s\varepsilon_{t-s}$ is a strictly stationary and ergodic process and $\hat{\delta}$ is the estimated coefficient of a time trend in the regression of S_{t-1} on（1, t）.　In the equations above, we should notice that our definitions of S_{t-1}^{τ}, g_{t-1}, and h_{t-1} slightly differ from those of Leybourne and Newbold (2000) because we regress y_{t-1} on (1, t) whereas they regress y_t on (1, t) to obtain the residuals.

f_0, f_1, and f_2 are defined based on Leybourne and Newbold (2000) as follows:

$$f_0 = \sum_{t=1}^{T} e_t^2,\ f_1 = \sum_{t=1}^{T} e_{t-1}^2,\ f_2 = \sum_{t=1}^{T} e_t e_{t-1}.$$

Using these three variables, $\hat{\sigma}^2$ and $\hat{\rho}$ are expressed as

$$\hat{\sigma}^2 = T^{-1}(f_0 + \hat{\rho}^2 f_1 - 2\hat{\rho} f_2),\ \hat{\rho} = f_2 f_1^{-1}.$$

1. Proof of Theorem 1 (a).

The test statistics are written as

$$T^{-1/2} t = T^{-1}(f_2 - f_1)(\hat{\sigma}^2 T^{-1} f_1)^{-1/2}, \tag{1a}$$

$$T^{-1}\{T(\hat{\rho}-1)\} = T^{-1}(f_2 - f_1)(T^{-1} f_1)^{-1}. \tag{2a}$$

To derive the limiting distributions of the statistics, we consider the probability limits of the three terms: $T^{-1}(f_2 - f_1)$, $T^{-1} f_1$, and $\hat{\sigma}^2$. The first term is

$$T^{-1}(f_2 - f_1) = T^{-1}\sum_{t=1}^{T} e_{t-1}\Delta e_t = T^{-1}\sum_{t=1}^{T} S_{t-1}^{\tau}\Delta S_t^{\tau} + T^{-1}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})\Delta(g_t - h_t)$$
$$+ T^{-1}\sum_{t=1}^{T} S_{t-1}^{\tau}\Delta(g_t - h_t) + T^{-1}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})\Delta S_t^{\tau}.$$

In the last equation, the terms are

$$T^{-1}\sum S_{t-1}^{\tau}\Delta S_t^{\tau} \xrightarrow{p} -\sigma^2(1+\rho)^{-1},\ T^{-1}\sum(g_{t-1} - h_{t-1})\Delta(g_t - h_t) = O(T^{-1}),$$
$$T^{-1}\sum S_{t-1}^{\tau}\Delta(g_t - h_t) = O(T^{-1}),\ T^{-1}\sum(g_{t-1} - h_{t-1})\Delta S_t^{\tau} = O_p(T^{-1}),$$

where $\xrightarrow{p}$ denotes convergence in probability.　Hence, we obtain

$$T^{-1}(f_2 - f_1) \xrightarrow{p} -\sigma^2(1+\rho)^{-1}. \tag{3a}$$

Next, $T^{-1}f_1$ is

$$T^{-1}f_1 = T^{-1}\sum_{t=1}^{T} e_{t-1}^2 = T^{-1}\sum_{t=1}^{T}(S_{t-1}^{\tau})^2 + T^{-1}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})^2 + 2T^{-1}\sum_{t=1}^{T} S_{t-1}^{\tau}(g_{t-1} - h_{t-1}).$$

The first and third terms in the last equation are

$$T^{-1}\sum (S_{t-1}^{\tau})^2 \xrightarrow{p} \sigma^2(1-\rho^2)^{-1}, 2T^{-1}\sum S_{t-1}^{\tau}(g_{t-1} - h_{t-1}) \xrightarrow{p} 0,$$

and assuming that

$$T^{-1}\sum (g_{t-1} - h_{t-1})^2 \to b_{11}.$$

Thus,

$$T^{-1}f_1 \xrightarrow{p} \sigma^2(1-\rho^2)^{-1} + b_{11}. \tag{4a}$$

For $\hat{\sigma}^2$, it can be written as

$$\begin{aligned}
\hat{\sigma}^2 &= T^{-1}\sum_{t=1}^{T}\Delta e_t^2 - T^{-1}(f_2 - f_1)^2 f_1^{-1} \\
&= T^{-1}\sum_{t=1}^{T}(\Delta S_t^{\tau})^2 + T^{-1}\sum_{t=1}^{T}\left[\Delta(g_t - h_t)\right]^2 + 2T^{-1}\sum_{t=1}^{T}\Delta S_t^{\tau}\Delta(g_t - h_t) \\
&\qquad - T^{-2}(f_2 - f_1)^2(T^{-1}f_1)^{-1}.
\end{aligned}$$

Using the facts that:

$$T^{-1}\sum (\Delta S_t^{\tau})^2 \xrightarrow{p} 2\sigma^2(1+\rho)^{-1}, T^{-1}\sum\left[\Delta(g_t - h_t)\right]^2 = O(T^{-1}),$$

$$2T^{-1}\sum \Delta S_t^{\tau}\Delta(g_t - h_t) = O_p(T^{-1}),$$

then,

$$\hat{\sigma}^2 \xrightarrow{p} 2\sigma^2(1+\rho)^{-1} - \left\{-\sigma^2(1+\rho)^{-1}\right\}^2 \left\{\sigma^2(1-\rho^2)^{-1} + b_{11}\right\}^{-1}$$
$$= \sigma^2\left\{\sigma^2 + 2(1-\rho)b_{11}\right\}\left\{\sigma^2 + (1-\rho^2)b_{11}\right\}^{-1}. \qquad (5a)$$

Proof of b_{11}.

$$T^{-1}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})^2 = T^{-1}\sum g_{t-1}^2 + T^{-1}\sum h_{t-1}^2 - 2T^{-1}\sum g_{t-1}h_{t-1}.$$

The first term in the right hand side is obtained as

$$T^{-1}\sum g_{t-1}^2 = T^{-1}\sum(d_{t-1}^2 + \bar{d}^2 - 2d_{t-1}\bar{d})$$
$$\to k_1^2\tau_1(1-\tau_1) + k_2^2\tau_2(1-\tau_2) + 2k_1k_2\tau_1(1-\tau_2)$$

by using

$$T^{-1}\sum d_{t-1}^2 \to k_1^2(1-\tau_1) + k_2^2(1-\tau_2) + 2k_1k_2(1-\tau_2).$$

The second term is

$$T^{-1}\sum h_{t-1}^2 \to 3\left\{k_1\tau_1(1-\tau_1) + k_2\tau_2(1-\tau_2)\right\}^2,$$

and the last term is

$$2T^{-1}\sum g_{t-1}h_{t-1} = 2T^{-1}\sum(d_{t-1} - \bar{d})(t-\bar{t})\sum(t-\bar{t})d_{t-1}\left\{\sum(t-\bar{t})^2\right\}^{-1}$$
$$= 2\left\{T^{-2}\sum d_{t-1}(t-\bar{t})\right\}^2\left\{T^{-3}\sum(t-\bar{t})^2\right\}^{-1} \to 6\left\{k_1\tau_1(1-\tau_1) + k_2\tau_2(1-\tau_2)\right\}^2.$$

Thus, we obtain

$$b_{11} = k_1^2\tau_1(1-\tau_1)(3\tau_1^2 - 3\tau_1 + 1) + k_2^2\tau_2(1-\tau_2)(3\tau_2^2 - 3\tau_2 + 1)$$
$$+ 2k_1k_2\tau_1(1-\tau_2)(3\tau_1\tau_2 - 3\tau_2 + 1). \quad \blacksquare$$

2. Proof of Theorem 1（b）.

The test statistics are given by

$$T^{-1/2}t = T^{-2}(f_2 - f_1)(\hat{\sigma}^2 T^{-3} f_1)^{-1/2}, \tag{6a}$$

$$T(\hat{\rho}-1) = T^{-2}(f_2 - f_1)(T^{-3} f_1)^{-1}. \tag{7a}$$

The probability limits of $T^{-2}(f_2 - f_1)$, $T^{-3} f_1$, and $\hat{\sigma}^2$ are derived first, and then, the proof of b_{21}, b_{22}, and b_{23} is given.　The term $T^{-2}(f_2 - f_1)$ is

$$T^{-2}(f_2 - f_1) = T^{-2}\sum_{t=1}^{T} S_{t-1}^{\tau}\Delta S_t^{\tau} + T^{-2}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})\Delta(g_t - h_t)$$
$$+ T^{-2}\sum_{t=1}^{T} S_{t-1}^{\tau}\Delta(g_t - h_t) + T^{-2}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})\Delta S_t^{\tau}.$$

Utilizing the facts that

$$T^{-2}\sum S_{t-1}^{\tau}\Delta S_t^{\tau} = O_p(T^{-1}),\ T^{-2}\sum(g_{t-1} - h_{t-1})\Delta h_t = 0,$$
$$T^{-2}\sum S_{t-1}^{\tau}\Delta(g_t - h_t) = O_p(T^{-1}),\ T^{-2}\sum(g_{t-1} - h_{t-1})\Delta S_t^{\tau} = O_p(T^{-1}),$$

and supposing that

$$T^{-2}\sum(g_{t-1} - h_{t-1})\Delta g_t \to b_{21},$$

then, we obtain

$$T^{-2}(f_2 - f_1) \xrightarrow{p} b_{21}. \tag{8a}$$

Now, we show

$$T^{-3} f_1 = T^{-3}\sum_{t=1}^{T}(S_{t-1}^{\tau})^2 + T^{-3}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})^2 + 2T^{-3}\sum_{t=1}^{T} S_{t-1}^{\tau}(g_{t-1} - h_{t-1}).$$

In the equation above,

$$T^{-3}\sum(S_{t-1}^{\tau})^2 = O_p(T^{-2}), 2T^{-3}\sum S_{t-1}^{\tau}(g_{t-1}-h_{t-1}) = O_p(T^{-3/2}),$$

and we write

$$T^{-3}\sum(g_{t-1}-h_{t-1})^2 \to b_{23}.$$

Thus,

$$T^{-3}f_1 \xrightarrow{p} b_{23}. \tag{9a}$$

Finally,

$$\begin{aligned}\hat{\sigma}^2 &= T^{-1}\sum_{t=1}^{T}\Delta e_t^2 - T^{-4}(f_2-f_1)^2(T^{-3}f_1)^{-1} \\ &= T^{-1}\sum_{t=1}^{T}(\Delta S_t^{\tau})^2 + T^{-1}\sum_{t=1}^{T}\left[\Delta(g_t-h_t)\right]^2 + 2T^{-1}\sum_{t=1}^{T}\Delta S_t^{\tau}\Delta(g_t-h_t) \\ &\qquad - T^{-4}(f_2-f_1)^2(T^{-3}f_1)^{-1}.\end{aligned}$$

We use the following relationships.

$$T^{-1}\sum(\Delta S_t^{\tau})^2 \xrightarrow{p} 2\sigma^2(1+\rho)^{-1}, 2T^{-1}\sum\Delta S_t^{\tau}\Delta(g_t-h_t) = O_p(T^{-1}).$$

And we let

$$T^{-1}\sum\left[\Delta(g_t-h_t)\right]^2 \to b_{22}.$$

So, the limit is

$$\hat{\sigma}^2 \xrightarrow{p} 2\sigma^2(1+\rho)^{-1} + b_{22} - \frac{b_{21}^2}{b_{23}}. \tag{10a}$$

Proof of b_{21}.

$$T^{-2}\sum_{t=1}^{T}(g_{t-1}-h_{t-1})\Delta g_t = T^{-2}\sum(g_{t-1}-h_{t-1})\Delta d_t = T^{-2}\sum g_{t-1}\Delta d_t - T^{-2}\sum h_{t-1}\Delta d_t.$$

The limit of the first term in the last equation is

$$T^{-2}\sum g_{t-1}\Delta d_t \to 2^{-1}\left\{k_1^2\tau_1(1-\tau_1)^2 + k_2^2\tau_2(1-\tau_2)^2 + k_1k_2(1-\tau_1)(1-\tau_2)(\tau_1+\tau_2)\right\}.$$

Note that

$$\sum DT_{t-1}^1\cdot DU_t^1 = \sum DT_{t-1}^1, \sum DT_{t-1}^1\cdot DU_t^2 = \sum_{t=1}^{T-\tau_2T}(\tau_2T-\tau_1T-1+t),$$
$$\sum DT_{t-1}^2\cdot DU_t^1 = \sum DT_{t-1}^2, \sum DT_{t-1}^2\cdot DU_t^2 = \sum DT_{t-1}^2.$$

And then,

$$T^{-2}\sum h_{t-1}\Delta d_t = \left\{T^{-2}\sum(t-\bar{t})\Delta d_t\right\}\left\{T^{-3}\sum(t-\bar{t})d_{t-1}\right\}\left\{T^{-3}\sum(t-\bar{t})^2\right\}^{-1}$$
$$\to 2^{-1}\left\{k_1\tau_1(1-\tau_1)+k_2\tau_2(1-\tau_2)\right\}\left\{k_1(1-\tau_1)^2(2\tau_1+1)+k_2(1-\tau_2)^2(2\tau_2+1)\right\}$$

using the following limits.

$$T^{-2}\sum(t-\bar{t})\Delta d_t \to 2^{-1}\left\{k_1\tau_1(1-\tau_1)+k_2\tau_2(1-\tau_2)\right\},$$
$$T^{-3}\sum(t-\bar{t})d_{t-1} \to 12^{-1}\left\{k_1(1-\tau_1)^2(2\tau_1+1)+k_2(1-\tau_2)^2(2\tau_2+1)\right\},$$
$$T^{-3}\sum(t-\bar{t})^2 \to 12^{-1}.$$

Thus,

$$b_{21} = 2^{-1}\left\{k_1^2\tau_1^2(1-\tau_1)^2(2\tau_1-1)+k_2^2\tau_2^2(1-\tau_2)^2(2\tau_2-1)\right.$$
$$\left.+2k_1k_2\tau_1\tau_2(1-\tau_1)(1-\tau_2)(\tau_1+\tau_2-1)\right\}.$$

Proof of b_{22}.

$$T^{-1}\sum_{t=1}^{T}\left[\Delta(g_t-h_t)\right]^2 = T^{-1}\sum\Delta d_t^2 + T^{-1}\sum\Delta h_t^2 - 2T^{-1}\sum\Delta d_t\Delta h_t.$$

All the terms in the right hand side are

$$T^{-1}\sum \Delta d_t^2 = T^{-1}\sum\left[k_1 DU_t^1 + k_2 DU_t^2\right]^2 \to k_1^2(1-\tau_1)+k_2^2(1-\tau_2)+2k_1k_2(1-\tau_2),$$
$$T^{-1}\sum \Delta h_t^2 \to \left\{k_1(1-\tau_1)^2(2\tau_1+1)+k_2(1-\tau_2)^2(2\tau_2+1)\right\}^2,$$
$$2T^{-1}\sum \Delta d_t \Delta h_t$$
$$\to 2\left\{k_1(1-\tau_1)^2(2\tau_1+1)+k_2(1-\tau_2)^2(2\tau_2+1)\right\}\left\{k_1(1-\tau_1)+k_2(1-\tau_2)\right\}.$$

Hence, we obtain

$$b_{22} = k_1^2\tau_1(1-\tau_1)\left\{\tau_1^2(2\tau_1-3)(1-2\tau_1)+(1-\tau_1)(2\tau_1+1)\right\}$$
$$+k_2^2\tau_2(1-\tau_2)\left\{\tau_2^2(2\tau_2-3)(1-2\tau_2)+(1-\tau_2)(2\tau_2+1)\right\}$$
$$+2k_1k_2\tau_1(1-\tau_2)\left\{\tau_1\tau_2(2\tau_1-3)(1-2\tau_2)+(1-\tau_2)(2\tau_2+1)\right\}.$$

Proof of b_{23}.

$$T^{-3}\sum_{t=1}^{T}(g_{t-1}-h_{t-1})^2 = T^{-3}\sum g_{t-1}^2 + T^{-3}\sum h_{t-1}^2 - 2T^{-3}\sum g_{t-1}h_{t-1}.$$

Since the limits of the terms in the equation above are

$$T^{-3}\sum g_{t-1}^2 \to 12^{-1}\left\{k_1^2(1-\tau_1)^3(3\tau_1+1)+k_2^2(1-\tau_2)^3(3\tau_2+1)\right.$$
$$\left.+2k_1k_2(1-\tau_2)^2(-3\tau_1^2+2\tau_2+1)\right\},$$
$$T^{-3}\sum h_{t-1}^2 \to 12^{-1}\left\{k_1(1-\tau_1)^2(2\tau_1+1)+k_2(1-\tau_2)^2(2\tau_2+1)\right\}^2$$
$$2T^{-3}\sum g_{t-1}h_{t-1} \to 6^{-1}\left\{k_1(1-\tau_1)^2(2\tau_1+1)+k_2(1-\tau_2)^2(2\tau_2+1)\right\}^2,$$

b_{23} is written as

$$b_{23} = 3^{-1}\left\{k_1^2\tau_1^3(1-\tau_1)^3 + k_2^2\tau_2^3(1-\tau_2)^3 - k_1k_2\tau_1^2(1-\tau_2)^2(2\tau_1\tau_2+\tau_1-3\tau_2)\right\}. \quad \blacksquare$$

3. Proof of Theorem 3.

The test statistics are written as

$$t = T^{-1}(f_2 - f_1)(\hat{\sigma}^2 T^{-2} f_1)^{-1/2}, \tag{11a}$$

$$T(\hat{\rho} - 1) = T^{-1}(f_2 - f_1)(T^{-2} f_1)^{-1}. \tag{12a}$$

We first show $T^{-1}(f_2 - f_1)$, $T^{-2} f_1$, and $\hat{\sigma}^2$, and then we prove c_{11}, c_{12}, and c_{22}.

$$T^{-1}(f_2 - f_1) = T^{-1}\sum_{t=1}^{T} S_{t-1}^{\tau}\Delta S_t^{\tau} + T^{-1}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})\Delta(g_t - h_t)$$
$$+ T^{-1}\sum_{t=1}^{T} S_{t-1}^{\tau}\Delta(g_t - h_t) + T^{-1}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})\Delta S_t^{\tau}.$$

It is straightforward to show that

$$T^{-1}\sum S_{t-1}^{\tau}\Delta S_t^{\tau} \xrightarrow{p} -\sigma^2(1+\rho)^{-1}, T^{-1}\sum(g_{t-1} - h_{t-1})\Delta h_t = 0,$$
$$T^{-1}\sum S_{t-1}^{\tau}\Delta(g_t - h_t) = O_p(T^{-1/2}), T^{-1}\sum(g_{t-1} - h_{t-1})\Delta S_t^{\tau} = O_p(T^{-1/2}).$$

Assuming that

$$T^{-1}\sum(g_{t-1} - h_{t-1})\Delta g_t \to c_{i1},$$

then, we obtain

$$T^{-1}(f_2 - f_1) \xrightarrow{p} -\sigma^2(1+\rho)^{-1} + c_{i1},\ i = 1,2. \tag{13a}$$

Next, we show

$$T^{-2} f_1 = T^{-2}\sum_{t=1}^{T}(S_{t-1}^{\tau})^2 + T^{-2}\sum_{t=1}^{T}(g_{t-1} - h_{t-1})^2 + 2T^{-2}\sum_{t=1}^{T} S_{t-1}^{\tau}(g_{t-1} - h_{t-1}).$$

The first and third terms in the right hand side are

$$T^{-2}\sum(S_{t-1}^{\tau})^2 = O_p(T^{-1}),\ 2T^{-2}\sum S_{t-1}^{\tau}(g_{t-1}-h_{t-1}) = O_p(T^{-1}),$$

and letting

$$T^{-2}\sum(g_{t-1}-h_{t-1})^2 \to c_{i3},$$

then, we find that

$$T^{-2}f_1 \xrightarrow{p} c_{i3},\ i=1,2. \tag{14a}$$

Now, we prove

$$\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\Delta e_t^2 + O_p(T^{-1})$$
$$= T^{-1}\sum_{t=1}^{T}(\Delta S_t^{\tau})^2 + T^{-1}\sum_{t=1}^{T}[\Delta(g_t-h_t)]^2 + 2T^{-1}\sum_{t=1}^{T}\Delta S_t^{\tau}\Delta(g_t-h_t) + O_p(T^{-1}).$$

The first and third terms are

$$T^{-1}\sum(\Delta S_t^{\tau})^2 \xrightarrow{p} 2\sigma^2(1+\rho)^{-1},\ 2T^{-1}\sum\Delta S_t^{\tau}\Delta(g_t-h_t) \xrightarrow{p} 0.$$

And let

$$T^{-1}\sum[\Delta(g_t-h_t)]^2 \to c_{i2}.$$

Thus,

$$\hat{\sigma}^2 \xrightarrow{p} 2\sigma^2(1+\rho)^{-1} + c_{i2},\ i=1,2. \tag{15a}$$

Proof of c_{11}.

$$T^{-1}\sum_{t=1}^{T}(g_{t-1}-h_{t-1})\Delta g_t = T^{-1}\sum g_{t-1}\Delta d_t - T^{-1}\sum h_{t-1}\Delta d_t.$$

The first term of the right hand side is

$$T^{-1}\sum g_{t-1}\Delta d_t = k_1k_2 - (k_1+k_2)\{k_1(1-\tau_1)+k_2(1-\tau_2)\}.$$

In this derivation, note that

$$\sum DU_{t-1}^1\cdot(DU_t^2 - DU_{t-1}^2) = \sum DU_{t-1}^1\cdot D(\tau_2T)_t = 1,$$

where $D(\tau_2T)_t = 1$ if $t=\tau_2T+1$ and 0 otherwise.　And then,

$$T^{-1}\sum h_{t-1}\Delta d_t \to 3\{k_1(2\tau_1-1)+k_2(2\tau_2-1)\}\{k_1\tau_1(1-\tau_1)+k_2\tau_2(1-\tau_2)\}$$

using the facts that

$$T^{-3/2}\sum(t-\bar{t})\Delta d_t \to 2^{-1}\{k_1(2\tau_1-1)+k_2(2\tau_2-1)\},$$
$$T^{-5/2}\sum(t-\bar{t})d_{t-1} \to 2^{-1}\{k_1\tau_1(1-\tau_1)+k_2\tau_2(1-\tau_2)\}, T^{-3}\sum(t-\bar{t})^2 \to 12^{-1}.$$

Thus,

$$c_{11} = -k_1^2(1-\tau_1)(6\tau_1^2-3\tau_1+1) - k_2^2(1-\tau_2)(6\tau_2^2-3\tau_2+1)$$
$$-k_1k_2\{(1-\tau_1)(6\tau_1\tau_2-3\tau_1+1)+(1-\tau_2)(6\tau_1\tau_2-3\tau_2+1)-1\}.$$

Proof of c_{12}.

$$T^{-1}\sum_{t=1}^{T}[\Delta(g_t-h_t)]^2 = T^{-1}\sum\Delta g_t^2 + T^{-1}\sum\Delta h_t^2 - 2T^{-1}\sum\Delta g_t\Delta h_t.$$

The term $T^{-1}\sum\Delta g_t^2$ is

$$T^{-1}\sum\Delta g_t^2 = T^{-1}\sum\Delta d_t^2 = \sum\{k_1D(\tau_1T)_t + k_2D(\tau_2T)_t\}^2 = k_1^2+k_2^2,$$

where $D(\tau_1T)_t = 1$ if $t=\tau_1T+1$ and 0 otherwise.　And then,

$$T^{-1}\sum \Delta h_t^2 = O(T^{-1}),\, 2T^{-1}\sum \Delta g_t \Delta h_t = O(T^{-1}).$$

Hence,

$$c_{12} = k_1^2 + k_2^2.$$

For the proof of c_{22}, since $\sum_{t=1}^{T}\left[\Delta(g_t - h_t)\right]^2$ is $O(1)$ for shifts in slope with $Scale = T^{-1/2}$, $T^{-1}\sum_{t=1}^{T}\left[\Delta(g_t - h_t)\right]^2 \to 0$ as $T \to \infty$; therefore $c_{22} = 0$. ■